

INDIAN STATISTICAL INSTITUTE

Mid-Semester Examination (First Semester) : 2026-2027

Course Name : M. MATH.

Subject name : MATHEMATICAL LOGIC

Date : September 12, 2026

Maximum Marks : 40

Duration : 2 hours

Answer all questions. Notations are as used in the class.

1. Let L be a sound and complete first-order logic. Show that L is compact. [5]
2. Let $\mathbb{N}$ be the set of natural numbers. Let L be a first-order language with equality together with a predicate symbol $<$, constant symbols 0 , 1 , and function symbols $+$ and $\times$, which are interpreted in $\mathbb{N}$ in the usual way. Using compactness theorem, show that there is a model of $\text{Th}(\mathbb{N})$, containing a number c , say, which is larger than any element of $\mathbb{N}$. [6]
3. Let $\mathcal{R}$ be the structure $(\mathbb{R}; +_{\mathbb{R}}, \cdot_{\mathbb{R}})$, $\mathbb{R}$ being the set of real numbers, $+$ and $\cdot$ correspond to the usual addition and multiplication, respectively. Let $\mathcal{L}$ be a first order language with equality, two binary function symbols $+$ and $\cdot$, interpreted as $+$ and $\cdot$, respectively in the structure $\mathcal{R}$. A formula $\varphi(x)$ in $\mathcal{L}$ with exactly one free variable x is said to define a set S of real numbers if $(s \in S \text{ iff } \varphi(s) \text{ holds in } \mathcal{R} \text{ with } x \text{ assigned to } s)$.
 - (a) Give a formula in $\mathcal{L}$ that defines the set $\{x \in \mathbb{R} : x \geq 0\}$ in the structure $\mathcal{R}$.
 - (b) Give a formula in $\mathcal{L}$ that defines the set $\{0\}$ in the structure $\mathcal{R}$. [5]
4. Check whether the following formulas are valid: [5]
 - (a) $(\exists x \varphi \vee \psi) \leftrightarrow \exists x(\varphi \vee \psi)$, x is not free in ψ .
 - (b) $\forall x(\varphi \rightarrow \psi) \leftrightarrow (\varphi \rightarrow \forall x \psi)$, x is not free in φ .
5. Using Natural Deduction or Sequent Calculus, show the following: [3 + 3]
 - (a) $\exists x \forall y \varphi \rightarrow \forall y \exists x \varphi$ is a theorem in FOL.
 - (b) $(\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma))$ is a theorem in CPL
6. Consider 3 suspects of committing a crime: A, B and C. At least one of them is guilty but not all. If A is guilty, then C is too. If B is innocent, then so is C. Can B be guilty? Justify your answer using classical propositional reasoning. [3]
7. Assume that a first order language has a two place predicate symbol P . Consider a formula φ in this language, which is the conjunction of the following formulas:
 - (a) $\forall x \exists y Pxy$
 - (b) $\forall x \neg Pxx$
 - (c) $\forall x \forall y \forall z ((Pxy \wedge Pyz) \rightarrow Pxz)$
 - Can φ have a finite model? Justify your answer. [4]
 - Does your conclusion still hold if you drop any one of the conjuncts from the conjunction? [6]

Natural Deduction Rules (FOL):

Implication:

$$\frac{\begin{array}{c} \overline{A}^1 \\ \vdots \\ B \end{array}}{A \rightarrow B}^1 \rightarrow I \qquad \frac{A \rightarrow B \quad A}{B} \rightarrow E$$

Conjunction:

$$\frac{A \quad B}{A \wedge B} \wedge I \qquad \frac{A \wedge B}{A} \wedge E_l \qquad \frac{A \wedge B}{B} \wedge E_r$$

Negation:

$$\frac{\begin{array}{c} \overline{A}^1 \\ \vdots \\ \perp \end{array}}{\neg A}^1 \neg I \qquad \frac{\neg A \quad A}{\perp} \neg E$$

Disjunction:

$$\frac{A}{A \vee B} \vee I_l \qquad \frac{B}{A \vee B} \vee I_r \qquad \frac{\begin{array}{cc} \overline{A}^1 & \overline{B}^1 \\ \vdots & \vdots \\ A \vee B & C \end{array}}{C}^1 \vee E$$

Truth and falsity:

$$\frac{\perp}{A} \perp E \qquad \frac{}{\top} \top I$$

Bi-implication:

$$\frac{\begin{array}{c} \overline{A}^1 \quad \overline{B}^1 \\ \vdots \quad \vdots \\ B \quad A \end{array}}{A \leftrightarrow B}^1 \leftrightarrow I \qquad \frac{A \leftrightarrow B \quad A}{B} \leftrightarrow E_l \qquad \frac{A \leftrightarrow B \quad B}{A} \leftrightarrow E_r$$

Reductio ad absurdum (proof by contradiction):

$$\frac{\begin{array}{c} \overline{\neg A}^1 \\ \vdots \\ \perp \end{array}}{A}^1 \text{ RAA}$$

The universal quantifier:

$$\frac{A(x)}{\forall y A(y)} \forall I \qquad \frac{\forall x A(x)}{A(t)} \forall E$$

In the introduction rule, x should not be free in any uncanceled hypothesis. In the elimination rule, t can be any term that does not clash with any of the bound variables in A .

The existential quantifier:

$$\frac{A(t)}{\exists x A(x)} \exists I \qquad \frac{\begin{array}{c} \overline{A(y)}^1 \\ \vdots \\ \exists x A(x) \quad B \end{array}}{B}^1 \exists E$$

In the introduction rule, t can be any term that does not clash with any of the bound variables in A . In the elimination rule, y should not be free in B or any uncanceled hypothesis.

Sequent Calculus Rules (FOL):

Identity rules:

$$\text{init} \frac{}{\varphi \Rightarrow \varphi} \quad \text{cut} \frac{\Gamma \Rightarrow \Delta, \varphi \quad \varphi, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

Left structural rules:

$$\text{w}_l \frac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta} \quad \text{c}_l \frac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta} \quad \text{ex}_l \frac{\Gamma, \psi, \varphi, \Gamma' \Rightarrow \Delta}{\Gamma, \varphi, \psi, \Gamma' \Rightarrow \Delta}$$

Right structural rules:

$$\text{w}_r \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi} \quad \text{c}_r \frac{\Gamma \Rightarrow \Delta, \varphi, \varphi}{\Gamma \Rightarrow \Delta, \varphi} \quad \text{ex}_r \frac{\Gamma \Rightarrow \Delta, \psi, \varphi, \Delta'}{\Gamma \Rightarrow \Delta, \varphi, \psi, \Delta'}$$

Left introduction rules:

$$\perp_l \frac{}{\perp, \Gamma \Rightarrow \Delta} \quad \neg_l \frac{\Gamma \Rightarrow \Delta, \varphi}{\neg \varphi, \Gamma \Rightarrow \Delta} \quad \vee_l \frac{\varphi, \Gamma \Rightarrow \Delta \quad \psi, \Gamma \Rightarrow \Delta}{\varphi \vee \psi, \Gamma \Rightarrow \Delta} \quad \wedge_l^0 \frac{\varphi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta} \quad \wedge_l^1 \frac{\psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta}$$

Right introduction rules:

$$\top_r \frac{}{\Gamma \Rightarrow \Delta, \top} \quad \neg_r \frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg \varphi} \quad \vee_r^0 \frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi} \quad \vee_r^1 \frac{\Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi} \quad \wedge_r \frac{\Gamma \Rightarrow \Delta, \varphi \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}$$

Left quantifier introduction rules:

$$\forall_l \frac{\varphi[t/x], \Gamma \Rightarrow \Delta}{\forall x \varphi, \Gamma \Rightarrow \Delta} \quad \exists_l \frac{\varphi[y/x], \Gamma \Rightarrow \Delta}{\exists x \varphi, \Gamma \Rightarrow \Delta} \quad y \text{ is fresh}$$

Right quantifier introduction rules:

$$\forall_r \frac{\Gamma \Rightarrow \Delta, \varphi[y/x]}{\Gamma \Rightarrow \Delta, \forall x \varphi} \quad y \text{ is fresh} \quad \exists_r \frac{\Gamma \Rightarrow \Delta, \varphi[t/x]}{\Gamma \Rightarrow \Delta, \exists x \varphi}$$

the side condition “ y is fresh” means that y is not free in any of the formulas in the conclusion of the rule.